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Fractional operators and some special functions☆

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This paper is dedicate to Prof. José Rodríguez Expósito on the occasion of his 60th birthday

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ABSTRACT

This paper considers the Riemann–Liouville fractional operator as a tool to reduce linear ordinary equations with variable coefficients to simpler problems, avoiding the singularities of the original equation. The main result is that this technique allow us to obtain an extension of the classical integral representation of the special functions related with the original differential equations. In particular, we will use as examples the cases of the well-known Generalized, Gauss and Confluent Hypergeometric equations, Laguerre equation, Hermite equation, Legendre equation and Airy equation.

1. Introduction

This paper is devoted to showing the efficiency of the called Riemann–Liouville fractional operators to reduce linear differential equations with variable coefficients to more elementary equations and to avoid the singularities of such equations. So, making changes of variables that implies fractional operators, we can represent the more frequently used special functions in terms of basic functions and then we can obtain solutions of the original equations in a simpler form. This representation will also allow us, in many cases, to extend the integral representation of the referred special functions.

We shall consider, by way of example, some differential operators and its relation to the Riemann–Liouville ones, as well as the solution to the differential equations of Laguerre, Hermite, Legendre, Airy and also Generalized, Gauss and Confluent hypergeometric equations.

As is well known the classical method for solving the aforementioned differential equations involved using series with indeterminate coefficients, when searching for a solution around a regular point, and the Frobenius method, when obtaining solutions around regular singular points. This latter technique implies considering several particular cases, depending on the roots of the characteristic equation.

This paper is organized as follows: in Section 2, which has an introductory nature, we introduce certain definitions and properties of the Riemann–Liouville operators and prove a set of properties, which will be useful in the development of the paper, to relate the referred operators with the $\delta = xD$ operator, where D represents the usual differential operator. The key results are shown in Section 3, where we introduce the method and apply it to some differential equations and obtain a generalization of the integral representation of several known special functions. Also, numerical approach of the mentioned integral representation is computed.

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2. Preliminary results

In this section we introduce some fractional operators, along with a set of properties that will be of use as we proceed in our discussion (see, for example, [1–4]).

Definition 1 (*Riemann–Liouville Fractional Operator*). Let $\alpha > 0$, with $n-1 < \alpha < n$ and $n \in \mathbb{N}$, $[a,b] \subset \mathbb{R}$ and let f be a suitable real function (for example, it suffices if $f \in L_1(a,b)$). The following definitions are well known, if $x > a$:

$$(I_{a+}^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\mathrm{d}t \tag{1}$$

$$(D_{a+}^{\alpha} f)(x) = D^n (I_{a+}^{n-\alpha} f)(x), \tag{2}$$

and if $x < b$:

$$(I_{b-}^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)\mathrm{d}t \tag{3}$$

$$(D_{b-}^{\alpha} f)(x) = D^n (I_{b-}^{n-\alpha} f)(x), \tag{4}$$

where D is the usual differential operator.

Definition 2 (*Generalized Riemann–Liouville Operators*). Under the same conditions for function f as in the above definitions, let g be a real function such that its derivative $g'(x)$ on $[a,b]$ is greater than 0. Then, if $x > a$:

$$(I_{a+;\,g}^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{g'(t)f(t)}{(g(x)-g(t))^{1-\alpha}}\mathrm{d}t \tag{5}$$

$$(D_{a+;\,g}^{\alpha} f)(x) = D_g^n (I_{a+;\,g}^{n-\alpha} f)(x) \tag{6}$$

and if $x < b$:

$$(I_{b-;\,g}^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \frac{g'(t)f(t)}{(g(t)-g(x))^{1-\alpha}}\mathrm{d}t \tag{7}$$

$$(D_{b-;\,g}^{\alpha} f)(x) = (-1)^n D_g^n (I_{b-;\,g}^{n-\alpha} f)(x), \tag{8}$$

where $D_g^n = \left(\frac{1}{g'(x)} D\right)^n$.

In particular, for the function $g(x) = x^m$, $m \in \mathbb{N}$, and $a = 0$ we obtain the following fractional integral operators, for $x > 0$:

$$\begin{aligned}(I_m^{\alpha} f)(x) &= (I_{0+;\,x^m}^{\alpha} f)(x) \\ &= \frac{1}{\Gamma(\alpha)} \int_0^x (x^m - t^m)^{\alpha-1} f(t)\mathrm{d}t^m \\ &= \frac{x^{m\alpha}}{\Gamma(\alpha)} \int_0^1 (1-z^m)^{\alpha-1} f(xz)\mathrm{d}z^m\end{aligned} \tag{9}$$

$$(D_m^{\alpha} f)(x) = (D_{0+;\,x^m}^{\alpha} f)(x) = (-D_m^1)^n (I_m^{n-\alpha} f)(x). \tag{10}$$

Then, if $\alpha = 1$ and $x > 0$, we have:

$$(I_m^1 f)(x) = \int_0^x f(t)\mathrm{d}t^m = m \int_0^x f(t)\, t^{m-1}\mathrm{d}t \tag{11}$$

$$m(D_m^1 f)(x) = x^{-m}(\delta f)(x) \tag{12}$$

where we are using the well-known notation $\delta = xD$.

Therefore, the following relations hold:

$$(D_m^1 I_m^1 f)(x) = f(x) \tag{13}$$

$$(I_m^1 D_m^1 f)(x) = \int_0^x \frac{t^{-m}}{m}\, (\delta f)(t)\mathrm{d}t^m = f(x) - f(0). \tag{14}$$

The following two Properties are well known:

Property 1. *Let f be a suitable function (for instance, locally integrable or continuous) and $\alpha, \beta > 0$. Then the following relations hold:*

$$(I_{a+}^{\alpha} I_{a+}^{\beta} f)(x) = (I_{a+}^{\alpha+\beta} f)(x) \tag{15}$$

$$(I_m^{\alpha} I_m^{\beta} f)(x) = (I_m^{\alpha+\beta} f)(x). \tag{16}$$

Property 2. *Let $\beta > -1$ and $\alpha > 0$ $(n-1 < \alpha < n, n \in \mathbb{N})$. Then*

$$D_m^{\alpha} x^{m\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}\, x^{m(\beta-\alpha)}. \tag{17}$$

Next, we introduce certain essential properties to proving the key result obtained in this paper.

Property 3. *Let f be a differentiable real function of order 1 in a certain interval $I \subset \mathbb{R}$, $\gamma \in \mathbb{R}$, and $m \in \mathbb{N}$. Then:*

$$(\delta\, x^{\gamma}) f(x) = (x^{\gamma}(\gamma+\delta)) f(x). \tag{18}$$

Property 4. *Let f be a differentiable real function of order 2 in a certain interval $I \subset \mathbb{R}$. Then:*

$$D^2 f(x) = (x^{-2}\delta(\delta-1)) f(x). \tag{19}$$

Property 5. *Let f be a differentiable real function of order 2 in a certain interval $I \subset \mathbb{R}$ and let a, b be constants. Then:*

$$(\delta+a)(\delta+b) f = (\delta+b)(\delta+a) f = (x^2 D^2 + (a+b+1)xD + ab) f. \tag{20}$$

Property 6. *Let $\alpha > 0$, $m \in \mathbb{N}$, and let f be a suitable function in a certain interval $I \subset \mathbb{R}$ (for instance, $f \in C^1(I)$, that is, differentiable with a continuous derivative). Then:*

$$(I_m^{\alpha}\delta) f(x) = ((\delta - m\alpha) I_m^{\alpha}) f(x) \tag{21}$$

and

$$(D_m^{\alpha}\delta) f(x) = ((\delta + m\alpha) D_m^{\alpha}) f(x) \tag{22}$$

where I_m^{α} and D_m^{α} are the Riemann–Liouville fractional operators in (9) and (10), respectively.

Property 7. *Under the same hypothesis as in the above property, the following relations hold:*

$$(I_m^{\alpha} x^{-m}\delta) f(x) = (x^{-m}\delta I_m^{\alpha})(f(x) - f(0)) \tag{23}$$

and

$$(D_m^{\alpha} x^{-m}\delta) f(x) = (x^{-m}\delta D_m^{\alpha})(f(x) - f(0)) \tag{24}$$

where I_m^{α} is the fractional Riemann–Liouville operator in (9).

In particular, for $f(0) = 0$ it holds that

$$(x^{-m}\delta I_m^{\alpha}) f(x) = (I_m^{\alpha} x^{-m}\delta) f(x). \tag{25}$$

Property 8. *Let $\gamma > 0$ $(n-1 < \alpha < n, n \in \mathbb{N})$ and $m \in \mathbb{N}$. Then, $D_m^{\alpha}\psi(x) = 0$ if, and only if:*

$$\psi(x) = \sum_{k=1}^{n} C_k\, x^{m(\alpha-k)}. \tag{26}$$

The proofs of Properties 3–6, can be find in the preprint [5].

Now, we introduce a new operator that we use in the following theorem:

Lemma 1. *Let f be a differential function of order 2 in a certain interval $I \subset \mathbb{R}$, with $x \neq 0$. Let us consider the differential operator*

$$L_\alpha = D^2 + \frac{D}{x} - \frac{\alpha^2}{x^2}, \tag{27}$$

then:

$$L_\alpha f = x^{-2}(\delta - \alpha)(\delta + \alpha)f. \tag{28}$$

Next, we present a commutative property applicable to differential operators. This rule will serve as a tool for reducing differential equations with variable coefficients to other more basic equations:

Theorem 1. *Let f be a differential function of order 2 in a certain interval $I \subset \mathbb{R}$, $x \neq 0$, L_α the operator (27) and T^α the operator given by*

$$T^\alpha = x^\alpha D_2^{\alpha+\frac{1}{2}} \quad (\alpha \in \mathbb{R},\ n = -[-\alpha],\ n \in \mathbb{N}), \tag{29}$$

where $D_2^{\alpha+\frac{1}{2}}$ is the Riemann–Liouville fractional differential operator in (8) and $[x]$ denotes the integer part of x. Then the following relationship holds:

$$(T^\alpha D^2)f(x) = (L_\alpha T^\alpha)(f(x) - f(0)). \tag{30}$$

Proof. Use Lemma 1 – formula (28) – and Properties 3, 4, 6 and 7. □

This theorem can be find in the preprint [5].

3. Reduction of differential equations by means of fractional tool

In this section we apply the previous results to explicitly obtain the solutions to a group of equations. We shall also show that in many cases one of the solutions extends the well-known integral representation of some special functions (see, for example, Abramowitz and Stegun [6]).

Remark 1. When using the fractional operators it is necessary to consider that one derivative from negative order corresponds to an integral of positive order and vice versa.

Remark 2. The numerical results that we will show were obtained by means of the function 'quadl' of MATLAB, which allows the numerical approximation of integrals using Gauss–Legendre polynomials.

3.1. Generalized hypergeometric equation

Let us consider the Generalized Hypergeometric Equation in terms of the δ operator:

$$x^{-1}\delta(\delta + c_1 - 1)\cdot\dots\cdot(\delta + c_q - 1)y - (\delta + a_1)\cdot\dots\cdot(\delta + a_p)y = 0 \tag{31}$$

where p and q are natural numbers.

In first place, let us study the special case

$$x^{-1}\delta(\delta + c - 1)y - y = 0 \tag{32}$$

whose first solution is the well-known hypergeometric function ${}_0F_1(c, x)$.

This equation can be written in terms of the L_α operator given in (27), with $\alpha = 1 - c$, introducing some changes of variables. One of them is to substitute $x^{-\frac{c-1}{2}}z$ for y in (32):

$$x^{-1}\delta(\delta + c - 1)x^{-\frac{c-1}{2}}z - x^{-\frac{c-1}{2}}z = 0 \tag{33}$$

$$x^{-1}\left(\delta - \frac{c-1}{2}\right)\left(\delta + \frac{c-1}{2}\right)z - z = 0. \tag{34}$$

Now, apply other change of variable, by putting $x = \frac{t^2}{4}$, and define $\delta_t = tD_t = t\frac{\mathrm{d}}{\mathrm{d}t}$, obtaining:

$$t^{-2}(\delta_t + (1 - c))(\delta_t - (1 - c))z - z = 0 \tag{35}$$

$$(L_\alpha - 1)z(t) = 0. \tag{36}$$

So, making the substitution $z(t) = T^{\alpha}(u(t) - u(0))$ in this equation yields:

$$(L_{\alpha} - 1)z(t) = T^{\alpha}[(D^2 - 1)u(t) - u(0)] = 0, \tag{37}$$

since

$$\begin{aligned}(L_{\alpha} - 1)z(t) &= ((L_{\alpha} - 1)T^{\alpha})(u(t) - u(0))\\ &= (L_{\alpha}T^{\alpha} - T^{\alpha})(u(t) - u(0))\\ &= T^{\alpha}[(D^2 - 1)u(t) - u(0)].\end{aligned}$$

Therefore, we can obtain a solution to the original equation at every point $x = x_0$ (unless $x_0 = 0$, in which case we can obtain a solution in any neighborhood of $x_0 = 0$), by simply choosing a solution to the basic differential equation:

$$(D^2 - 1)u(t) = u(0), \tag{38}$$

that is,

$$u(t) = C_1(\exp(t) - \exp(-t)) + u(0) \quad (C_1 \text{ a real constant}) \tag{39}$$

and taking $C_1 = \frac{1}{2}$ we obtain

$$u(t) = \sinh(t) + u(0). \tag{40}$$

We shall choose the solution $u_1(t) = \sinh(t)$, which yields the following solutions for the original equation, valid for any real value of α:

$$z_1(t) = t^{\alpha} D_2^{\alpha+\frac{1}{2}} \sinh(t) \tag{41}$$

$$= t^{\alpha} D_2^{\alpha+\frac{1}{2}} I_2^1 D_2^1 \sinh(t) \tag{42}$$

$$= \frac{t^{\alpha}}{2} D_2^{\alpha-\frac{1}{2}} t^{-1} \cosh(t) \tag{43}$$

$$= \frac{t^{1-c}}{2} D_2^{\frac{1}{2}-c} t^{-1} \cosh(t). \tag{44}$$

So, for $c > \frac{1}{2}$ we obtain the classical representation of the solution:

$$\begin{aligned}z_1(t) &= \frac{t^{1-c}}{2} I_2^{c-\frac{1}{2}} t^{-1} \cosh(t)\\ &= \frac{t^{c-1}}{\Gamma\left(c - \frac{1}{2}\right)} \int_0^1 (1 - z^2)^{c-\frac{3}{2}} \cosh(tz)\mathrm{d}z.\end{aligned}$$

Therefore,

$$y_1(x) = \frac{2^{c-1}}{\Gamma\left(c - \frac{1}{2}\right)} \int_0^1 (1 - z^2)^{c-\frac{3}{2}} \cosh(2\sqrt{x}z)\mathrm{d}z. \tag{45}$$

Aside from a constant, expression (45) represents the classical integral representation of the generalized hypergeometric function:

$$y_1(x) = \frac{\sqrt{\pi}}{2^{2-c}\Gamma(c)} {}_0F_1(c, x). \tag{46}$$

Finally, for $c < \frac{1}{2}$ we obtain a generalization of the hypergeometric function ${}_0F_1$:

$$z_1(t) = \frac{t^{1-c}}{2} D_2^{\frac{1}{2}-c} t^{-1} \cosh(t).$$

Then, if $n = -[c - 1/2]$ (we use the notation [.] to the integer part of the argument):

$$\begin{aligned}y_1(x) &= \frac{x^{1-c}}{2^c} \left[D_2^{\frac{1}{2}-c} t^{-1} \cosh(t) \right]_{t=2\sqrt{x}}\\ &= \frac{x^{1-c}}{2^c} \left[D_2^n I_2^{n+c-\frac{1}{2}} t^{-1} \cosh(t) \right]_{t=2\sqrt{x}}\\ &= \frac{\sqrt{\pi}}{2^{2-c}\Gamma(c)} {}_0F_1(c, x). \qquad (47)\end{aligned}$$

In particular, if $n = 1$, that is $-\frac{1}{2} < c < \frac{1}{2}$, then

$$ {}_0F_1(c, x) = \frac{2\Gamma(c)}{\sqrt{\pi}\,\Gamma(c + \frac{1}{2})} \int_0^1 (1 - z^2)^{c+\frac{1}{2}} \left[\sqrt{x}z \sinh(2\sqrt{x}z) + c \cosh(2\sqrt{x}z)\right] \mathrm{d}z. \tag{48} $$

In order to illustrate the exactitude of this integral representation, we use the approach of Gauss–Legendre to calculate the function ${}_0F_1$ and to compare it with the exact value. We show these data in the following table:

	${}_0F_1\,(1.7, 2)$	${}_0F_1\,(0.2, 2)$
Exact value	2.6997	22.2932
Numeric value	2.6997	22.2932
Error	8.3875×10^{-8}	1.2578×10^{-6}

Lastly, let us point out that various procedures exist for obtaining a linearly independent second solution $y_2(x)$ from $y_1(x)$ for the generalized hypergeometric equation (32). The most natural method is to reduce the original equation to a first order linear equation, for which we already know a solution, and then to solve that equation directly. Also, we could extend the used method for obtaining $y_1(x)$:

$$ y_2(x) = y_1(x) \int \frac{1}{x\,(y_1(x))^2} \mathrm{d}x \tag{49} $$

$$ = \left(\frac{\sqrt{\pi}}{\Gamma(b)}\, {}_0F_1(c, x)\right) \int \frac{1}{x\left(\frac{\sqrt{\pi}}{\Gamma(b)}\, {}_0F_1(c, x)\right)^2} \mathrm{d}x. \tag{50} $$

Additionally, keeping in mind the Property 8 (relation (26)), it directly holds that every solution to

$$ (D^2 - 1)u(t) = u(0) + t^{(2\alpha-1)}, \tag{51} $$

is also a solution to the generalized hypergeometric equation (36), except for $t = 0$.

Finding a particular solution to Eq. (51) we obtain:

$$ u_p(t) - u_p(0) = \int_0^t s^{2\alpha-1} (\sinh(s) \cosh(t) - \cosh(s) \sinh(t)) \mathrm{d}s \tag{52} $$

therefore, a second linearly independent solution to the Bessel equation is:

$$ y_2(x) = (2x)^{1-c} \left[D_2^{\frac{3}{2}-c} (u_p(t) - u_p(0)) \right]_{t=2\sqrt{x}}. \tag{53} $$

Returning to the general case

$$ x^{-1}\delta(\delta + c_1 - 1) \cdot \cdots \cdot (\delta + c_q - 1)y - (\delta + a_1) \cdot \cdots \cdot (\delta + a_p)y = 0. \tag{54} $$

We can use fractional operators to introduce changes of variables and reduce the original equation:

- If $p \geq q$ we take

  $$ y = D^{a_p} x^{a_p+1-c_q} D^{a_{p-1}+1-c_q} x^{a_{p-1}+1-c_{q-1}} \ldots D^{a_{p-q+1}+1-c_2} x^{a_{p-q+1}+1-c_1} z $$

  and thus we reduce q orders the original equation.

  By example, let us consider the Generalized Hypergeometric Equation of order 3:

  $$ [x^{-1}\delta(\delta + c_1 - 1)(\delta + c_2 - 1) - (\delta + a_1)(\delta + a_2)(\delta + a_3)]y = 0. $$

  Then, applying the correspondent change of variable

  $$ y = D^{a_3} x^{a_3+1-c_2} D^{a_2+1-c_2} x^{a_2+1-c_1} z $$

  we can obtain a solution solving this equation of order 1:

  $$ x^{-1}\delta z - (\delta + a_1 - c_1 + 1)z = 0. $$

- If $p < q$ we take

  $$ y = D^{a_p} x^{a_p+1-c_q} D^{a_{p-1}+1-c_q} x^{a_{p-1}+1-c_{q-1}} \ldots D^{a_1+1-c_{q-p+2}} x^{a_1+1-c_{q-p+1}} z $$

  and thus we reduce p orders the original equation.

By example, let us consider the Generalized Hypergeometric Equation of order 4:

$$[x^{-1}\delta(\delta+c_1-1)(\delta+c_2-1)(\delta+c_3-1)-(\delta+a_1)(\delta+a_2)]y=0.$$

Then, applying the correspondent change of variable

$$y=D^{a_2}x^{a_2+1-c_3}D^{a_1+1-c_3}x^{a_1+1-c_2}z,$$

we obtain a solution solving this equation of order 2:

$$x^{-1}\delta(\delta+c_1-c_2)z-z=0.$$

Finally, using the obtained first solution, we can reduce the original equation and then we can obtain a linearly independent second solution.

3.2. Gauss hypergeometric equation

Using the fractional framework, we solve the Gauss Hypergeometric equation (55):

$$x(1-x)y''+[c-(a+b+1)x]y'-aby=0 \tag{55}$$

where a, b y c are real constants.

In first place, considering $x\neq 0$ ($x=0$ is a singular point of the equation), we regroup this equation and we use the Property 5 to obtain:

$$x^{-1}(x^2y''+cxy')-[x^2y''+(a+b+1)xy'+aby]=0$$
$$x^{-1}\delta(\delta+c-1)y-(\delta+a)(\delta+b)y=0.$$

Remark. Although $x=1$ is also other singular point, we get to avoid this singularity with the new change of variable that implies a fractional operator.

Now, we introduce the change of variable $y=x^{1-c}D^{b-c+1}z$ (keeping in mind that if $(b-c+1)<0$ then $D^{b-c+1}=I^{-b+c-1}$) and we apply the Properties 3, 6 and 7:

$$[x^{-1}\delta(\delta+c-1)-(\delta+a)(\delta+b)]x^{1-c}D^{b-c+1}z=0$$
$$[x^{-1}\delta(\delta+1-c)-(\delta+a+1-c)(\delta+b+1-c)]D^{b-c+1}z=0$$
$$[x^{-1}\delta D^{b-c+1}(\delta-b)-D^{b-c+1}(\delta+a-b)\delta]z=0$$
$$D^{b-c+1}[x^{-1}\delta(\delta-b)-(\delta+a-b)\delta]z+x^{-1}\delta D^{b-c+1}(\delta-b)z(0)=0.$$

If we suppose $z(0)=0$, then:

$$D^{b-c+1}[x^{-1}\delta(\delta-b)z-(\delta+a-b)\delta z]=0 \tag{56}$$

$$D^{b-c+1}[x^{-1}(\delta-b)\delta z-(\delta+a-b)\delta z]=0. \tag{57}$$

Considering $u=\delta z$ and applying the Property 8 (relation (26)), a solution of this equation is:

$$x^{-1}(\delta-b)u-(\delta+a-b)u=0. \tag{58}$$

Finally, making other change of variable, $u=x^b v$, and using the Property 3, we can reduce the original equation:

$$x^{-1}(\delta-b)x^b v-(\delta+a-b)x^b v=0 \tag{59}$$

$$x^{-1}x^b\delta v-x^b(\delta+a)v=0 \tag{60}$$

$$x^{-1}\delta v-(\delta+a)v=0 \tag{61}$$

$$(1-x)Dv-av=0. \tag{62}$$

Solving this basic differential equation,

$$v=c_0(1-x)^{-a} \tag{63}$$

where c_0 is a real constant, we obtain a solution of the Gauss Hypergeometric Equation:

$$y=c_0x^{1-c}D^{b-c}x^{b-1}(1-x)^{-a} \tag{64}$$

$$=c_0x^{1-c}D^nI^{n-b+c}x^{b-1}(1-x)^{-a} \tag{65}$$

$$=c_0x^{1-c}D^n\left[\frac{x^{n+c-1}}{\Gamma(n-b+c)}\int_0^1\frac{(1-t)^{n-b+c-1}}{(1-xt)^a}t^{b-1}\mathrm{d}t\right]$$

for $0 \leq n-1 < b-c < n$ with $n \in \mathbb{N}$, and

$$y = c_0 x^{1-c} I^{c-b} x^{b-1} (1-x)^{-a} \tag{66}$$

$$= c_0 \frac{1}{\Gamma(c-b)} \int_0^1 (1-t)^{c-b-1} t^{b-1} (1-tx)^{-a} \mathrm{d}t \tag{67}$$

for $b-c<0$.

Taking $c_0 = \frac{\Gamma(c)}{\Gamma(b)}$, this solution represents the Gauss Hypergeometric Function

$$y = {}_2F_1(a, b, c, x). \tag{68}$$

Now, using this solution, we can reduce the original equation and then we can obtain a linearly independent second solution.

In order to illustrate the exactitude of the obtained integral representation, we use the approach of Gauss–Legendre to calculate the function ${}_2F_1$ and to compare it with the exact value. We show these data in the following table:

	${}_2F_1$ (0.2, 1.7, 2, 2)	${}_2F_1$ (0.2, 1.5, 1, 0.5)
Exact value	0.9187 − 0.5782i	1.2506
Numeric value	0.9187 − 0.5782i	1.2506
Error	3.8282×10^{-5}	1.7139×10^{-7}

3.3. *Confluent hypergeometric equation*

Let us consider the Confluent Hypergeometric equation (69)

$$xy'' + (b-x)y' - ay = 0 \tag{69}$$

where a and b are real constants.

Reorganizing and using the Property 4, for $x \neq 0$, we can rewrite this equation:

$$x^{-1}(x^2D^2 + (b-x)xD)y - ay = 0 \tag{70}$$

$$x^{-1}\delta(\delta + b - 1)y - (\delta + a)y = 0. \tag{71}$$

Therefore, making a change of variable, $y = x^{1-b} D^{a-b} x^{a-1} z$ (where if $(a-b) < 0$ then $D^{a-b} = I^{-a+b}$), and applying the Properties 3, 6 and 7, we can reduce the original equation:

$$[x^{-1}\delta(\delta + b - 1) - (\delta + a)]x^{1-b} D^{a-b} x^{a-1} z = 0$$
$$[x^{-1}\delta(\delta + 1 - b) - (\delta + a + 1 - b)]D^{a-b} x^{a-1} z = 0$$
$$D^{a-b}[x^{-1}\delta(\delta + 1 - a) - (\delta + 1)]x^{a-1} z = 0$$
$$D^{a-b+1} I[(\delta + 1)x^{-1}(\delta + 1 - a) - (\delta + 1)]x^{a-1} z = 0$$
$$D^{a-b+1} x[x^{-1}(\delta + 1 - a) - 1]x^{a-1} z = 0$$
$$D^{a-b+1} x^{a-1}\delta z - D^{a-b+1} x x^{a-1} z = 0$$
$$D^{a-b+1} x^{a}(Dz - z) = 0.$$

Thus, we only have to solve this equation and we will obtain a more simple equation. Thus, a solution of this equation is:

$$x^a(z' - z) = 0 \tag{72}$$

$$z' - z = 0. \tag{73}$$

Now, solving this equation, we have a solution of the Confluent Hypergeometric Equation:

$$y = c_0 x^{1-b} D^{a-b} x^{a-1} \mathrm{e}^x \tag{74}$$

$$= c_0 x^{1-b} D^n I^{n-a+b} x^{a-1} \mathrm{e}^x \tag{75}$$

$$= c_0 x^{1-b} D^n \left[\frac{x^{n+b-1}}{\Gamma n - a + b} \int_0^1 (1-t)^{n-a+b-1} t^{a-1} \mathrm{e}^{xt} \mathrm{d}t \right]$$

for $0 \leq n-1 < a-b < n$ with $n \in \mathbb{N}$, and

$$y = c_0 x^{1-b} I^{b-a} x^{a-1} \mathrm{e}^x \tag{76}$$

$$= c_0 \frac{1}{\Gamma(b-a)} \int_0^1 (1-t)^{b-a-1} t^{a-1} \mathrm{e}^{xt} \mathrm{d}t \tag{77}$$

for $a-b<0$, with c_0 a real constant.

Furthermore, if we consider $c_0 = \frac{\Gamma(b)}{\Gamma(a)}$, then this solution represents a Confluent Hypergeometric Function:

$$y = {}_1F_1(a, b, x). \tag{78}$$

Now, using this solution, we can reduce the original equation and then we can obtain a linearly independent second solution.

In order to illustrate the exactitude of the obtained integral representation, we use the approach of Gauss–Legendre to calculate the function ${}_1F_1$ and to compare it with the exact value. We show these data in the following table:

	${}_1F_1\,(1.7, 2, 2)$	${}_1F_1\,(2, 1.7, 2)$
Exact value	5.8585	9.4809
Numeric value	5.8582	9.4809
Error	2.8390×10^{-4}	4.4210×10^{-7}

3.4. *Laguerre equation*

The Laguerre equation (79)

$$xy'' + (\alpha + 1 - x)y' + \lambda y = 0 \tag{79}$$

where $\alpha, \lambda \in \mathbb{R}$, is a particular case of the Confluent Hypergeometric Equation

$$xy'' + (b - x)y' - ay = 0 \tag{80}$$

where $b = \alpha + 1$ and $a = -\lambda$.

Then, we can apply the same techniques that we use to solve the Confluent Hypergeometric Equation and thus we can reduce and solve the Laguerre Equation:

$$y = c_0 x^{-\alpha} D^{-\lambda-\alpha-1} x^{-\lambda-1} \mathrm{e}^x \tag{81}$$

$$= c_0 x^{-\alpha} D^n I^{n+\lambda+\alpha+1} x^{-\lambda-1} \mathrm{e}^x \tag{82}$$

$$= c_0 x^{-\alpha} D^n \left[\frac{x^{n+\alpha}}{\Gamma(n+\lambda+\alpha+1)} \int_0^1 \frac{(1-u)^{n+\lambda+\alpha}}{u^{\lambda+1}} \mathrm{e}^{xu} \mathrm{d}u \right]$$

for $0 \le n - 1 < -\lambda - \alpha - 1 < n$, and

$$y = c_0 x^{-\alpha} I^{\lambda+\alpha+1} x^{-\lambda-1} \mathrm{e}^x \tag{83}$$

$$= c_0 \frac{1}{\Gamma(\lambda+\alpha+1)} \int_0^1 (1-t)^{\lambda+\alpha} t^{-\lambda-1} \mathrm{e}^{xt} \mathrm{d}t \tag{84}$$

for $-\lambda - \alpha - 1 < 0$, where c_0 is a real constant.

Furthermore, for $c_0 = \frac{\Gamma(\alpha+1)}{\Gamma(-\lambda)}$, this solution represents a Confluent Hypergeometric Function:

$$y = {}_1F_1(-\lambda, \alpha + 1, x). \tag{85}$$

Again, using this solution, we can reduce the original equation and then we can obtain a linearly independent second solution.

3.5. *Hermite equation*

Let us consider the Hermite equation (86):

$$y'' - 2xy' + 2\lambda y = 0 \tag{86}$$

with $\lambda \in \mathbb{R}$, that can be written in the form:

$$(x^{-2}\delta(\delta - 1) - 2\delta + 2\lambda)y = 0 \tag{87}$$

If we make the change of variable $t = x^2$ and we define $\delta_t = tD_t = t\frac{\mathrm{d}}{\mathrm{d}t}$ then we obtain this equation:

$$t^{-1}\delta_t\left(\delta_t - \frac{1}{2}\right)y - \left(\delta_t - \frac{\lambda}{2}\right)y = 0 \tag{88}$$

that is other particular case of the Confluent Hypergeometric Equation in terms of δ:

$$x^{-1}\delta(\delta + b - 1)y - (\delta + a)y = 0 \tag{89}$$

where $b = \frac{1}{2}$ and $a = \frac{-\lambda}{2}$.

Then a solution of the Hermite Equation is:

$$y = \left[c_0 t^{\frac{1}{2}} D_t^{\frac{-\lambda-1}{2}} \{t^{\frac{-\lambda}{2}-1} \mathrm{e}^t\}\right]_{t=x^2} \tag{90}$$

$$= \left[c_0 t^{\frac{1}{2}} D_t^n I_t^{n+\frac{\lambda+1}{2}} \{t^{\frac{-\lambda}{2}-1} \mathrm{e}^t\}\right]_{t=x^2} \tag{91}$$

$$= \left[c_0 t^{\frac{1}{2}} D_t^n \left(\frac{t^{n-\frac{1}{2}}}{\Gamma\left(n+\frac{\lambda+1}{2}\right)} \int_0^1 \frac{(1-u)^{n+\frac{\lambda-1}{2}}}{u^{\frac{\lambda}{2}+1}} \mathrm{e}^{tu} \mathrm{d}u\right)\right]_{t=x^2} \tag{92}$$

$$= c_0 \frac{1}{2} D^n \left[\frac{x^{2n-1}}{\Gamma\left(n+\frac{\lambda+1}{2}\right)} \int_0^1 \frac{(1-u)^{n+\frac{\lambda-1}{2}}}{u^{\frac{\lambda}{2}+1}} \mathrm{e}^{x^2 u} \mathrm{d}u\right] \tag{93}$$

for $0 \le n-1 < \frac{-\lambda-1}{2} < n$, and

$$y = \left[c_0 t^{\frac{1}{2}} I_t^{\frac{\lambda+1}{2}} \{t^{\frac{-\lambda}{2}-1} \mathrm{e}^t\}\right]_{t=x^2} \tag{94}$$

$$= c_0 \frac{1}{\Gamma\left(\frac{\lambda+1}{2}\right)} \int_0^1 (1-u)^{\frac{\lambda-1}{2}} u^{\frac{-\lambda}{2}-1} \mathrm{e}^{x^2 u} \mathrm{d}u \tag{95}$$

for $\frac{-\lambda-1}{2} < 0$, where c_0 is a real constant.

Furthermore, for $c_0 = \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{-n}{2}\right)}$, this solution represents a Confluent Hypergeometric Function:

$$y = \left[{}_1F_1\left(\frac{-n}{2}, \frac{1}{2}, t\right)\right]_{t=x^2}. \tag{96}$$

Once again, using this solution, we can reduce the original equation and then we can obtain a linearly independent second solution.

3.6. Legendre equation

Let us consider the Legendre equation (97)

$$(1-x^2)y'' - 2xy' + \lambda(\lambda+1)y = 0 \tag{97}$$

with $n \in \mathbb{R}$, and let we write it in terms of δ operator:

$$(1-x^2)x^{-2}\delta(\delta-1)y - 2\delta y + \lambda(\lambda+1)y = 0 \tag{98}$$

$$x^{-2}\delta(\delta-1)y - \delta(\delta+1)y + \lambda(\lambda+1)y = 0. \tag{99}$$

Remark. $x = 1$ and $x = -1$ are singular points of this equation, but we get to avoid these singularities introducing a change of variable.

Now, introducing the change of variable $t = x^2$ and defining $\delta_t = tD_t = t\frac{\mathrm{d}}{\mathrm{d}t}$, we obtain a new equation:

$$t^{-1}\delta_t\left(\delta_t - \frac{1}{2}\right)y - \delta_t\left(\delta_t + \frac{1}{2}\right)y + \frac{\lambda}{2}\frac{\lambda+1}{2}y = 0 \tag{100}$$

$$t^{-1}\delta_t\left(\delta_t - \frac{1}{2}\right)y - \left(\delta_t - \frac{\lambda}{2}\right)\left(\delta_t + \frac{\lambda+1}{2}\right)y = 0 \tag{101}$$

that is a particular case of the Gauss Hypergeometric Equation in terms of δ

$$x^{-1}\delta(\delta+c-1)y - (\delta+a)(\delta+b)y = 0 \tag{102}$$

where $a = -\frac{\lambda}{2}$, $b = \frac{\lambda+1}{2}$ and $c = \frac{1}{2}$.

Then, applying the method that we used to solve the Gauss Hypergeometric Equation, we obtain a solution of the Legendre equation:

$$y = \left[c_0 t^{\frac{1}{2}} D_t^{\frac{\lambda}{2}} \left\{ t^{\frac{\lambda-1}{2}} (1-t)^{\frac{\lambda}{2}} \right\} \right]_{t=x^2} \tag{103}$$

$$= \left[c_0 t^{\frac{1}{2}} D_t^n I_t^{n-\frac{\lambda}{2}} \left\{ t^{\frac{\lambda-1}{2}} (1-t)^{\frac{\lambda}{2}} \right\} \right]_{t=x^2} \tag{104}$$

$$= \left[c_0 t^{\frac{1}{2}} D_t^n \left(\frac{t^{n-\frac{1}{2}}}{\Gamma\left(n-\frac{\lambda}{2}\right)} \int_0^1 \frac{(1-u)^{n-\frac{\lambda}{2}-1}}{(1-tu)^{-\frac{\lambda}{2}}} u^{\frac{\lambda-1}{2}} \mathrm{d}u \right) \right]_{t=x^2} \tag{105}$$

$$= c_0 \frac{1}{2} D^n \left[\frac{x^{2n-1}}{\Gamma\left(n-\frac{\lambda}{2}\right)} \int_0^1 \frac{(1-u)^{n-\frac{\lambda}{2}-1}}{(1-x^2u)^{-\frac{\lambda}{2}}} u^{\frac{\lambda-1}{2}} \mathrm{d}u \right] \tag{106}$$

for $0 \le n-1 < \frac{\lambda}{2} < n$, and

$$y = \left[c_0 t^{\frac{1}{2}} I_t^{-\frac{\lambda}{2}} \left\{ t^{\frac{\lambda-1}{2}} (1-t)^{\frac{\lambda}{2}} \right\} \right]_{t=x^2} \tag{107}$$

$$= c_0 \frac{1}{\Gamma\left(-\frac{\lambda}{2}\right)} \int_0^1 (1-u)^{-\frac{\lambda}{2}-1} u^{\frac{\lambda-1}{2}} (1-x^2u)^{\frac{\lambda}{2}} \mathrm{d}u \tag{108}$$

for $\frac{\lambda}{2} < 0$, where c_0 is a real constant.

Furthermore, considering $c_0 = \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)}$, this solution represents the Gauss Hypergeometric Function

$$y = \left[{}_2F_1 \left(-\frac{n}{2}, \frac{n+1}{2}, \frac{1}{2}, t \right) \right]_{t=x^2}. \tag{109}$$

Similarly, using this solution, we can reduce the original equation and then we can obtain a linearly independent second solution.

3.7. Airy equation

The Airy equation (110)

$$y'' - xy = 0 \tag{110}$$

can be written in terms of the δ operator, considering $x \neq 0$ and using the Property 4:

$$x^{-3}\delta(\delta-1)y + y = 0. \tag{111}$$

To solve this equation, we will introduce successive changes of variable. The first of them is $t = x^3$:

$$9t^{-1}\delta_t \left(\delta_t - \frac{1}{3} \right) y + y = 0. \tag{112}$$

Now, we make the change $y = D_t^{\frac{1}{6}} z$ and we will apply the Property 6, supposing $z(0) = 0$:

$$9t^{-1}\delta_t \left(\delta_t - \frac{1}{3} \right) D_t^{\frac{1}{6}} z + D_t^{\frac{1}{6}} z = 0 \tag{113}$$

$$D_t^{\frac{1}{6}} 9t^{-1}\delta_t \left(\delta_t - \frac{1}{2} \right) z + D_t^{\frac{1}{6}} z = 0. \tag{114}$$

Lastly, we take $t = 9s^2$:

$$D_t^{\frac{1}{6}} \frac{1}{4} s^{-2} \delta_s (\delta_s - 1) z + D_t^{\frac{1}{6}} z = 0 \tag{115}$$

$$D_t^{\frac{1}{6}} (D_s^2 z + 4z) = 0. \tag{116}$$

Using the Property 8, we can reduce the original equation to other more basic equation:

$$D_s^2 z + 4z = c_0 t^{\left(\frac{1}{6}-1\right)} = 9c_0 s^{2\left(\frac{1}{6}-k\right)} \tag{117}$$

where c_0 is a real constant that we can take $c_0 = 0$.

Therefore, solving this equation,

$$z = \sin(2s) \tag{118}$$

$$= \sin\left(\frac{2}{3}\sqrt{t}\right) \tag{119}$$

we finally obtain a solution of the Airy Equation:

$$y = \left[D_t^{\frac{1}{6}} \sin\left(\frac{2}{3}\sqrt{t}\right)\right]_{t=x^3} \tag{120}$$

$$= \left[D_t I_t^{\frac{5}{6}} \sin\left(\frac{2}{3}\sqrt{t}\right)\right]_{t=x^3} \tag{121}$$

$$= \left[D_t \left(\frac{t^{\frac{5}{6}}}{\Gamma\left(\frac{5}{6}\right)} \int_0^1 (1-u)^{-\frac{1}{6}} \sin\left(\frac{2}{3}\sqrt{tu}\right) \mathrm{d}u\right)\right]_{t=x^3} \tag{122}$$

$$= \frac{1}{3x^2} D \left(\frac{x^{\frac{5}{2}}}{\Gamma\left(\frac{5}{6}\right)} \int_0^1 (1-u)^{-\frac{1}{6}} \sin\left(\frac{2}{3}\sqrt{x^3 u}\right) \mathrm{d}u\right).$$

Now, using this solution, we can reduce the original equation and then we can obtain a linearly independent second solution.

3.8. Reduction of differential equations of order 2 with polynomial coefficients

In general, using fractional operators, we can reduce differential equations of order 2 with polynomial coefficients, this is:

$$(a_0 + a_1 x + a_2 x^2) y'' + (b_0 + b_1 x) y' + c_0 y = 0 \tag{123}$$

where a_0, a_1, a_2, b_0, b_1 and c_0 are real numbers, with $a_0, a_1, a_2 \neq 0$.

Thus, considering $x \neq 0$, we can write this equation in the form:

$$y'' + \frac{a_1}{a_0} x^{-1} \left(x^2 y'' + \frac{b_0}{a_1} x y'\right) + \frac{a_2}{a_0} \left(x^2 y'' + \frac{b_1}{a_2} x y' + \frac{c_0}{a_2} y\right) = 0. \tag{124}$$

This equation can be written in terms of the δ operator as follow:

$$[x^{-1}\delta x^{-1}\delta + A x^{-1}\delta(\delta + B) + C(\delta + E)(\delta + F)]y = 0 \tag{125}$$

where $A = \frac{a_1}{a_0}$, $B = \frac{b_0}{a_1} - 1$, $C = \frac{a_2}{a_0}$ and E, F are solutions of the system $EF = \frac{c_0}{a_2}$, $E + F + 1 = \frac{b_1}{a_2}$.

Therefore, we introduce the change of variable $y = D^{F-1} z$, with $z(0) = 0$, to reduce the original equation:

$$[x^{-1}\delta x^{-1}\delta + A x^{-1}\delta(\delta + B) + C(\delta + E)(\delta + F)] D^{F-1} z = 0$$
$$D^F [x^{-1}\delta z + A(\delta + B - F + 1) z + Cx(\delta + E - F + 1) z] = 0.$$

So, we obtain a solution of the original equation solving the reduced equation, which have an order smaller than the original equation:

$$x^{-1}\delta z + A(\delta + B - F + 1) z + Cx(\delta + E - F + 1) z = 0. \tag{126}$$

Now, using the obtained first solution, we can reduce the original equation and then we can obtain a linearly independent second solution.

4. Conclusions

We must point out that some authors have studied the generalizations of the classical special functions in the framework of the fractional calculus, and they have applied techniques of transmutation, such as those used in this paper to solve the hypergeometric equation related with the function ${}_0F_1$, to find the solution of certain fractional differential equations involving generalized Bessel differential operators. See, for instance, [4,7–9].

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